# Bridging Neural ODE and ResNet : A Formal Error Bound for Safety Verification

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## Background

We consider a neural ODE of the form:

$$\dot{x}(t) = \frac{dx(t)}{dt} = f(x(t)), \tag{1}$$

with state $x \in \mathbb{R}^n$, initial state $x(0) = u$, and vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ defined as a finite sequence of classical neural network layers. The state trajectories of (1) are defined based on the solution $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ of the corresponding initial value problem:

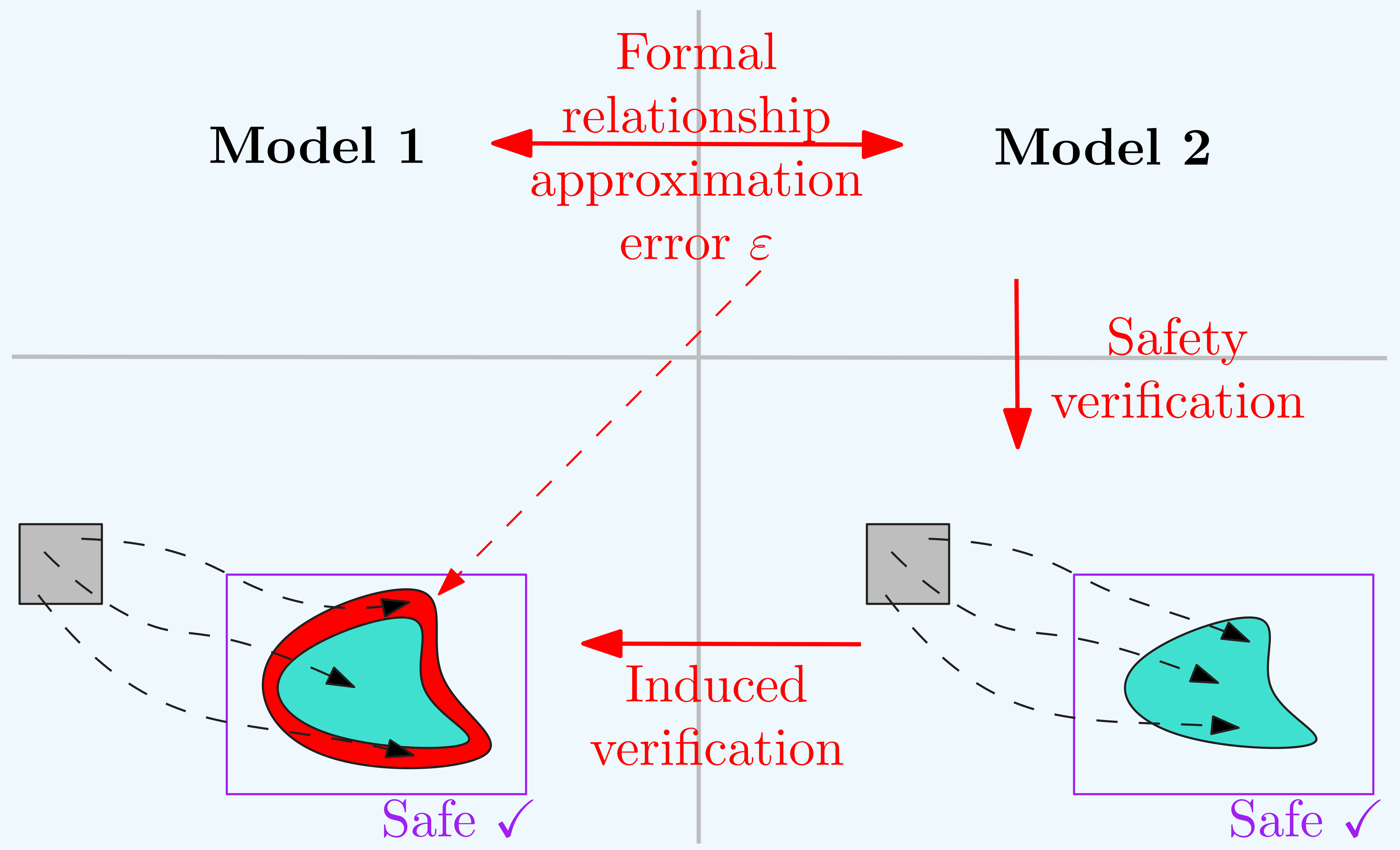
$$x(t) = \Phi(t, x(0)) = \Phi(t, u).$$

where, such a neural ODE is described as a continuous-depth generalization of a residual neural network constituted of a single residual block. Conversely, this ResNet can be seen as the Euler discretization of the neural ODE (1):

$$y = u + f(u), \tag{2}$$

## Verification Proxy

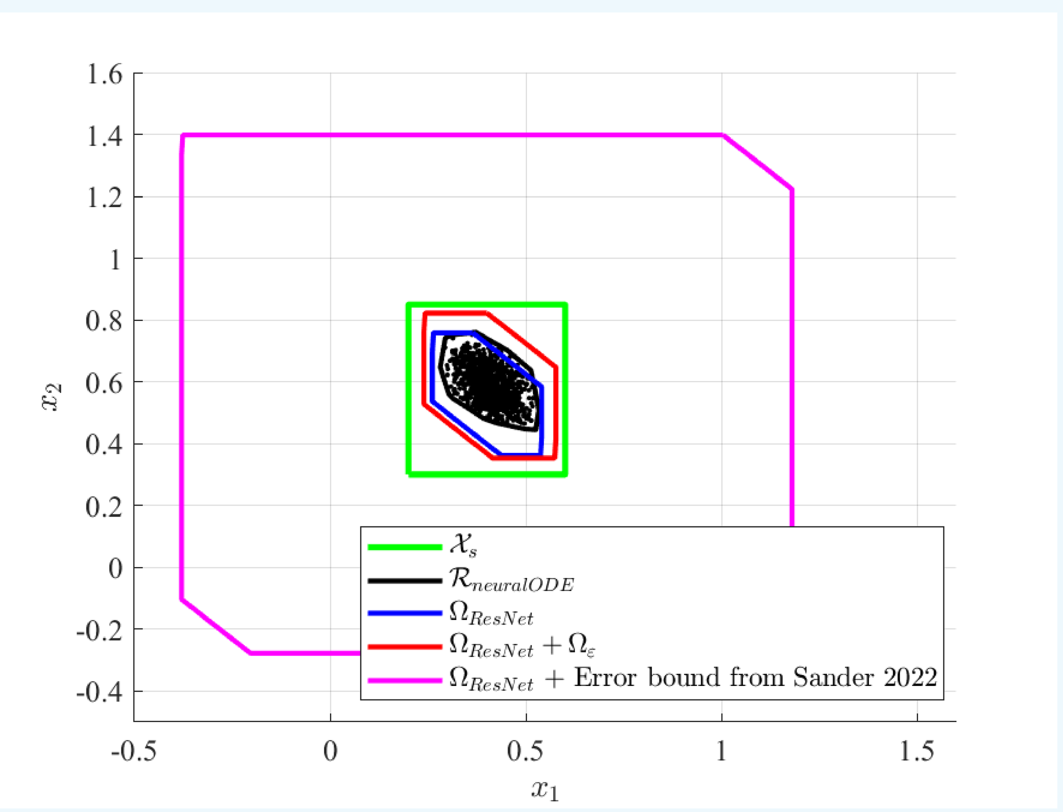
The proposed proxy verify the satisfaction of the safety property on one model by relying on the error set $\mathcal{R}_\varepsilon(\mathcal{X}_{in})$ from *Problem 1* and the reachability analysis of the other model.




## neural ODE based on ResNet

neural ODE true reachable set is guaranteed to be safe using the positive error bound $\Omega_\varepsilon$, as:

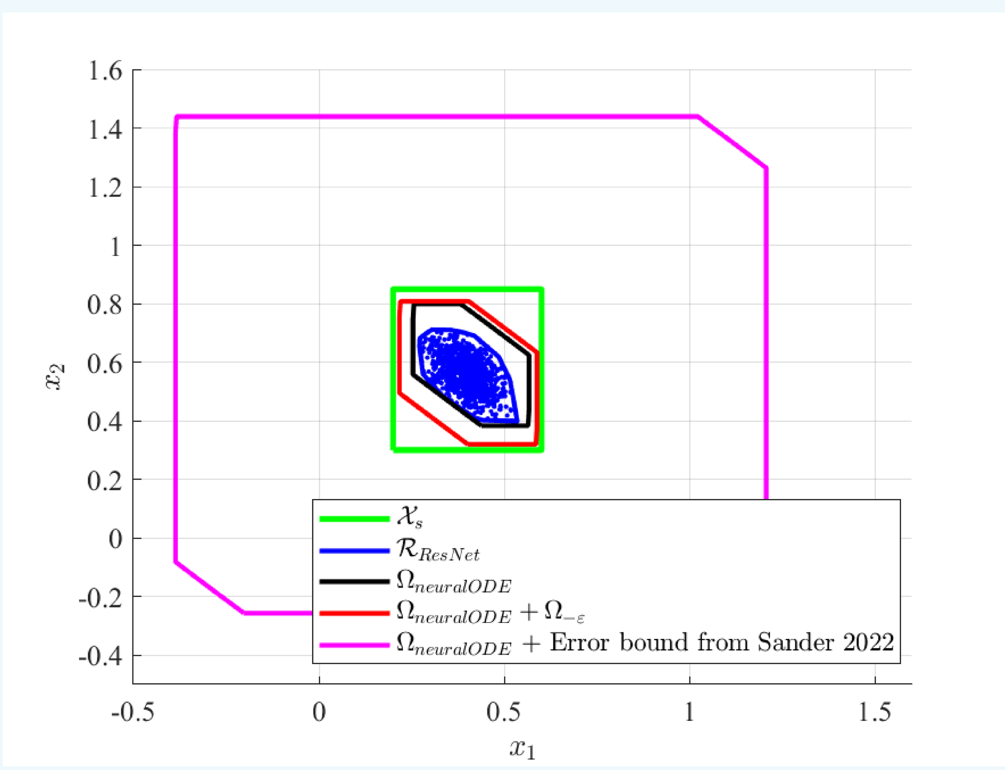
$$\mathcal{X}_s \supseteq \Omega_{\text{ResNet}}(\mathcal{X}_{in}) + \Omega_\varepsilon(\mathcal{X}_{in}) \supseteq \mathcal{R}_{\text{neural ODE}}.$$




## ResNet based on neural ODE

ResNet true reachable set is guaranteed to be safe using the negative error bound $\Omega_{-\varepsilon}$, as:

$$\mathcal{X}_s \supseteq \Omega_{\text{neural ODE}}(\mathcal{X}_{in}) + \Omega_{-\varepsilon}(\mathcal{X}_{in}) \supseteq \mathcal{R}_{\text{ResNet}}.$$




## Objectives

- Establish formal relations between discrete and continuous neural models, to deduce safety of one model based on the safety verification of the other.
- Analysis of the mathematical properties satisfied by the new continuous models.
- Exploiting these mathematical properties to study neural ODE behavior with respect to various features.

## Problem Definition

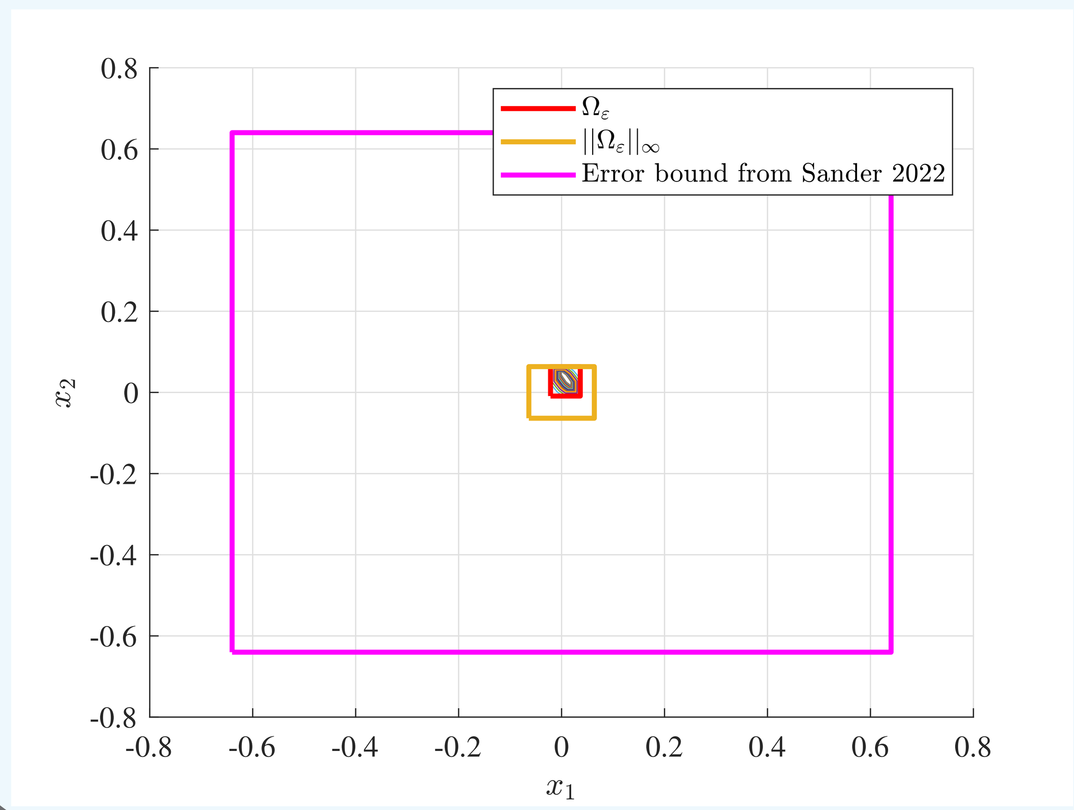
*Problem 1* (*Error Bounding*): For an input set $\mathcal{X}_{in} \subseteq \mathbb{R}^n$ we want to over-approximate the set $\mathcal{R}_\varepsilon(\mathcal{X}_{in})$ of errors between the ResNet (2) and neural ODE (1) models, defined as:

$$\mathcal{R}_\varepsilon(\mathcal{X}_{in}) = \{\Phi(1, u) - (u + f(u)) \mid u \in \mathcal{X}_{in}\}.$$

*Problem 2* (*Verification Proxy*): For an input-output safety property defined by an input set $\mathcal{X}_{in} \subseteq \mathbb{R}^n$ and a safe output set $\mathcal{X}_s \subseteq \mathbb{R}^n$, the verification problem consists in checking whether the reachable output set of a model is fully contained in the targeted safe set: $\mathcal{R}(\mathcal{X}_{in}) \subseteq \mathcal{X}_s$.

## Error Bound

The proposed error bound achieved tighter bounding compared to current SOTA [1] in magenta, where it was **10 times wider** (on each dimension) than the infinity norm of our error set in yellow, and about **16 millions times larger** (in volume over the 5-dimensional state space) than our error set $\Omega_\varepsilon(\mathcal{X}_{in})$ in red.




## Bibliography

[1] Sander, M., Ablin, P., Peyré, G.: Do residual neural networks discretize neural ordinary differential equations? Advances in Neural Information Processing Systems **35**, 36520–36532 (2022)

[2] Sayed, A.S., Meyer, P.J., Ghazel, M.: Bridging neural ode and resnet: A formal error bound for safety verification. In: International Symposium on AI Verification (2025), https://arxiv.org/abs/2506.03227